About the Group Inverse and Moore-Penrose Inverse of a Product

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ABSTRACT

Let A be a matrix over a ring. Suppose that P, Q, P', and Q' are matrices over the ring such that $P'PA = A = AQQ'$. If A has group inverse $A^{\#}$, then PAQ has a group inverse iff $AA^{\#}QPA + 1 - AA^{\#}$ is invertible. An analogous result for Moore-Penrose inverses is also given.

INTRODUCTION

In this paper we consider the group and Moore-Penrose inverses of a product PAQ. Specifically, we assume the existence of P' and Q' such that $P'PA = A = AQQ'$, and then characterize the existence of the group and Moore-Penrose inverses of PAQ in terms of the corresponding inverses of A. The results are then applied to several familiar factorizations of matrices: for example, the polar, the Schur, and the singular-value decompositions. For applications in the study of Hankel, Toeplitz, and Bezoutian matrices, the reader is referred to [4] and [5].

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1. RESULTS

If A is an $m \times n$ matrix over a ring, then the Moore-Penrose inverse $A^\dagger$ of A with respect to an involution $*$ is the unique solution, if it exists, of the system of equations $AXA = A$, $XAX = X$, $(AX)^* = AX$, and $(XA)^* = XA$. Also, if $m = n$, then the group inverse $A^\#$ of A is the unique solution, if it exists, of the system of equations $AXA = A$, $XAX = X$, and $AX = XA$. (See, for example, [1, pp. 7, 162].)

THEOREM 1. *Let A be an $n \times n$ matrix over a ring R with group inverse $A^\#$. If P, Q are matrices for which there exist matrices P' and Q' such that $P'PA = A$ and $AQQ' = A$, then the group inverse of the product PAQ exists iff $AA^\#QPA + 1_n - AA^\#$ is invertible. In this case,*

$$(PAQ)^\# = PA(AA^\#QPA + 1_n - AA^\#)^{-2}Q.$$

Proof. To simplify the notation, we denote PAQ by T. Suppose $T^\#$ exists. Since

$$T = TT^\#T, \qquad \text{then} \quad A = AQT^\#PA, \tag{1}$$

and since

$$TT^\# = T^\#T, \qquad \text{then} \quad AQT^\#PA = P'T^\#TPA.$$

Therefore, $A = P'T^\#TPA$, and

$$\begin{aligned} AA^\# = A^\#A &= A^\#P'T^\#TPA \\ &= \left[(AA^\#)A^\#P'T^\#PA(AA^\#)\right]\left[(AA^\#)QPA(AA^\#)\right]. \end{aligned}$$

Since $TT^\# = T^\#T$, then by (1),

$$AQTT^\#Q' = AQT^\#PA = A.$$

Multiplying this on the left by $A^\#$, we obtain

$$A^\#AQTT^\#Q' = A^\#A.$$

Consequently, since $A^{\#}A$ is idempotent,

$$AA^{\#} = \left[(AA^{\#})QPA(AA^{\#})\right]\left[(AA^{\#})QT^{\#}Q'(AA^{\#})\right].$$

This shows the invertibility of $(AA^{\#})QPA(AA^{\#})$ in the ring $AA^{\#}M_n(R)AA^{\#}$ and therefore the invertibility of $(AA^{\#})QPA(AA^{\#})+1_n - AA^{\#} = AA^{\#}QPA + 1_n - AA^{\#}$ in the ring $M_n(R)$. (See [6].)

Conversely, let $F = AA^{\#}$ and $Y = FQPA + 1 - F$. Then $F = F^2$, $AF = A$, and $YF = FQPAF + F - F^2 = FQPA = F^2QPA + F - F^2 = FY$. Hence, $AY^k = AFY^k = AY^kF$ for all integers k.

Now let $X = PAY^{-2}Q$. Then, $XPA = PAY^{-2}QPA = P(AY^{-2}F)QPA = PAY^{-2}(FQPA) = PAY^{-2}(YF) = PAY^{-1}F = PAY^{-1}$, $AQX = AFQPAY^{-2}Q = A(FQPA)Y^{-2}Q = A(FY)Y^{-2}Q = AY^{-1}Q = AY^{-1}FQ$, and

$$(AQX)PA = (AY^{-1}FQ)PA = AY^{-1}(FQPA) = AY^{-1}YF = A.$$

Therefore,

$$TXT = P(AQXPA)Q = T,$$

$$XTX = XP(AQX) = XP(AY^{-1}Q)$$

$$= (XPA)Y^{-1}Q = (PAY^{-1})Y^{-1}Q = PAY^{-2}Q = X,$$

and

$$TX = P(AQX) = P(AY^{-1}Q) = (PAY^{-1})Q$$

$$= (XPA)Q = XT.$$

That is, X is the group inverse of T. ∎

Corollary. *If $E = E^2$ and if P and Q are such that there exists a P' and Q' such that $P'PE = E = EQQ'$, then $(PEQ)^{\#}$ exists iff $EQPE + 1 - E$ is invertible. In this case,*

$$(PEQ)^{\#} = PE(EQPE + 1 - E)^{-2}Q.$$

Proof. If E is idempotent, then $E^{\#} = E$. ∎

REMARK. The existence of $(PAQ)^{\#}$ does not necessarily follow from the existence of $(PA)^{\#}$ and $(AQ)^{\#}$. This can be verified by taking

$$P=\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}, \qquad A=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \text{and} \quad Q=\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}.$$

THEOREM 2. *Let A be an $m\times n$ matrix over a ring R with Moore-Penrose inverse $A^{\dagger}$ with respect to an involution $*$ on the matrices over R. If PAQ is a product of matrices for which there exist matrices P' and Q' such that $P'PA=A$ and $AQQ'=A$, then the Moore-Penrose inverse of the product PAQ exists iff*

$$(PA)^*PA+1_n-A^{\dagger}A$$

and

$$(AQ)(AQ)^*+1_m-AA^{\dagger}$$

are invertible. In this case,

$$(PAQ)^{\dagger}=(AQ)^*\big[AQ(AQ)^*+1_m-AA^{\dagger}\big]^{-1}$$
$$\times A\big[(PQ)^*PA+1_n-A^{\dagger}A\big]^{-1}(PA)^*.$$

Proof. To simplify the notation, we denote PAQ by T. Suppose $T^{\dagger}$ exists, then since $T=TT^{\dagger}T$, then $A=AQT^{\dagger}PA$, and since $TT^{\dagger}=[TT^{\dagger}]^*$, then $AQT^{\dagger}PA=P'[TT^{\dagger}]^*PA$. Therefore, $A=P'T^{\dagger *}Q^*A^*P^*PA$, and $A^*=A^*P^*TT^{\dagger}P'^*$. This implies

$$\begin{aligned}A^{\dagger}A&=A^{\dagger}P'T^{\dagger *}Q^*A^*P^*PA\\ &=\big[(A^{\dagger}A)A^{\dagger}P'T^{\dagger *}Q^*(A^{\dagger}A)\big]\big[(A^{\dagger}A)A^*P^*PA(A^{\dagger}A)\big]\end{aligned}$$

and

$$\begin{aligned}A^*A^{\dagger *}&=A^*P^*TT^{\dagger}P'^*A^{\dagger *}\\ &=\big[(A^{\dagger}A)A^*P^*PA(A^{\dagger}A)\big]\big[(A^{\dagger}A)QT^{\dagger}P'^*A^{\dagger *}(A^{\dagger}A)\big].\end{aligned}$$

Consequently, $(A^{\dagger}A)A^*P^*PA(A^{\dagger}A)$ is invertible in the ring $A^{\dagger}AM_r(R)A^{\dagger}A$; therefore, $(PAA^{\dagger}A)^*(PAA^{\dagger}A)+1_n-A^{\dagger}A=(PA)^*PA+1_n-A^{\dagger}A$ is invertible in the ring $M_n(R)$.

It follows, analogously, from $A=AQT^{\dagger}PA$ and $T^{\dagger}T=(T^{\dagger}T)^*$ that $AQ(AQ)^*+1_m-AA^{\dagger}$ is invertible in the ring $M_m(R)$.

Conversely, let $G=AA^{\dagger}$, $H=A^{\dagger}A$, $R=AQ$, $S=PA$, $U=RR^*+1_m-G$, and $V=S^*S+1_n-H$. Then, since $G^2=G$ and $RR^*=GRR^*$, we have $RR^*U^{-1}=G(RR^*+1_m-G)U^{-1}=GUU^{-1}=G$. Analogously, since $H^2=H$ and $S^*S=S^*SH$, we have $V^{-1}S^*S=V^{-1}(S^*S+1_n-H)H=V^{-1}VH=H$.

Let $X=R^*U^{-1}AV^{-1}S^*$. Then

$$\begin{aligned}TX&=PAQX=PRR^*U^{-1}AV^{-1}S^*=P(RR^*U^{-1})AV^{-1}S^*\\&=PGAV^{-1}S^*=P(GA)V^{-1}S^*=PAV^{-1}S^*=(PA)V^{-1}S^*\\&=SV^{-1}S^*;\end{aligned}$$

hence $(TX)^*=TX$. Analogously, $XT=R^*U^{-1}R$ and $(XT)^*=XT$.

Moreover,

$$\begin{aligned}TXT&=T(XT)=PAQR^*U^{-1}R=PRR^*U^{-1}R=P(RR^*U^{-1})R\\&=PGR=PAA^{\dagger}AQ=T\end{aligned}$$

and

$$\begin{aligned}XTX&=(XT)X=R^*U^{-1}RX=R^*U^{-1}RR^*U^{-1}AV^{-1}S^*\\&=R^*U^{-1}(RR^*U^{-1})AV^{-1}S^*=R^*U^{-1}GAV^{-1}S^*=R^*U^{-1}(GA)V^{-1}S^*\\&=R^*U^{-1}AV^{-1}S^*=X.\end{aligned}$$

Consequently, X is the Moore-Penrose inverse of T. ∎

Corollary. *Under the conditions of Theorem 2 we also have*:

(1) $(PA)^{\dagger}=[(PA)^*PA+1_n-A^{\dagger}A]^{-1}(PA)^*$.
(2) $(AQ)^{\dagger}=(AQ)^*[AQ(AQ)^*+1_m-AA^{\dagger}]^{-1}$.
(3) $(PAQ)^{\dagger}=(AQ)^{\dagger}A(PA)^{\dagger}$.
(4) $(PAQ)^{\dagger}=Q^{\dagger}A^{\dagger}P^{\dagger}$, *under the additional assumption that* $(PA)^{\dagger}=A^{\dagger}P^{\dagger}$ *and* $(AQ)^{\dagger}=Q^{\dagger}A^{\dagger}$.

Proof. This follows from the theorem and the following facts: If $A^\dagger$ exists, then $(AA^*)^\dagger$ and $(A^*A)^\dagger$ exist,

$$(AA^* + 1_m - AA^\dagger)^{-1} = (AA^*)^\dagger + 1_m - AA^\dagger,$$

$$(A^*A + 1_n - A^\dagger A)^{-1} = (A^*A)^\dagger + 1_n - A^\dagger A,$$

$A^\dagger A$ commutes with $(PA)^*PA + 1_n - A^\dagger A$, and $AA^\dagger$ commutes with $AQ(AQ)^* + 1_m - AA^\dagger$. ■

REMARK. If an idempotent E has a Moore-Penrose inverse, and if P, Q, P', and Q' are such that $P'PE = E = EQQ'$, then $(PEQ)^\dagger$ can be characterized by Theorem 2. We claim that $E^\dagger$ exists w.r.t. an involution $*$ iff there exist matrices N and N' such that $E = (EN)N'$, $(EN)^* = EN = (EN)^2$, and $EE^* + 1 - EN$ is invertible. Indeed, if $E^\dagger$ exists, take $N = E^\dagger$ and $N' = E$. Conversely, if the conditions hold, it follows from Theorem 2 that $E^\dagger = E^*(EE^* + 1 - EN)^{-1}$.

Therefore, if $E^\dagger$ exists, $(PEQ)^\dagger$ exists iff $(PE)^*PE + 1 - E^\dagger E$ and $(EQ)(EQ)^* + 1 - EE^\dagger$ are invertible. In this case,

$$(PEQ)^\dagger = (EQ)^*[EQ(EQ)^* + 1 - EE^\dagger]^{-1}E[(PE)^*PE + 1 - E^\dagger E]^{-1}(PE)^*.$$

If $E^* = E$, which is the case in [6] for matrices over semisimple Artinian rings, then

$$(PEQ)^\dagger = (EQ)^*[EQ(EQ)^* + 1 - E]^{-1}[(PE)^*PE + 1 - E]^{-1}(PE)^*.$$

If in particular $E = 1$, then $(PQ)^\dagger$ exists iff P^*P and QQ^* are invertible. In this case, $(PQ)^\dagger = Q^*(QQ^*)^{-1}(P^*P)^{-1}P^*$. (See [7].)

2. APPLICATIONS

(1) Let M be an $m \times n$ matrix over a ring R. If M contains an invertible submatrix A, then the Schur decomposition of M by this matrix A is of the form

$$M = P^T\begin{pmatrix} 1 & 0 \\ CA^{-1} & 1 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix}\begin{pmatrix} 1 & A^{-1}B \\ 0 & 1 \end{pmatrix}Q^T,$$

where P and Q are permutation matrices. This means that if $(D-CA^{-1}B)^{\#}$ or $(D-CA^{-1}B)^{\dagger}$ is known, then $M^{\#}$ or $M^{\dagger}$ can be characterized by the theorems. If $(D-CA^{-1}B)^{\#}$ or $(D-CA^{-1}B)^{\dagger}$ is not known, the Schur decomposition can be applied again provided $D-CA^{-1}B$ contains an invertible submatrix. This process may be repeated. For an explicit example of a 5×5 matrix over the ring $\mathbf{Z}_2\oplus\mathbf{Z}_{3^2}$ of integers modulo 18, we refer the reader to [2].

(2) Let M be an $n\times n$ matrix over a field F. Then, by use of the Frobenius normal form, M is similar over F to the direct sum of the companion matrices C_i of the elementary divisors of $\lambda 1_n - M$; i.e.,

$$M = P^{-1}(C_1\oplus C_2\oplus\cdots\oplus C_k)P.$$

Clearly, the group inverse of $C_1\oplus C_2\oplus\cdots\oplus C_k$ exists iff for all $i=1,2,\ldots,k$, the group inverse of C_i exists. In this case, $(C_1\oplus C_2\oplus\cdots\oplus C_k)^{\#}C_k^{\#}\oplus\cdots\oplus C_k^{\#}$, and we can apply Theorem 1 to characterize the group inverse of M.

(3) Let $\bar{a}$ denote the complex conjugate of an element a in the field $\mathbf{C}$ of complex numbers. The unary operation $(a_{ij})\to(a_{ij})^* = (\bar{a}_{ji})$ on the set Mat($\mathbf{C}$) of all matrices over $\mathbf{C}$ is a well-known matrix involution. Let M be a nonzero $m\times n$ matrix of rank r over $\mathbf{C}$, and let $M^{\dagger}$ be its Moore-Penrose inverse with respect to the involution $*$.

(a) M has the factorization

$$M = P^T\begin{pmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}Q^T$$

$$= P^T\begin{pmatrix} 1_r\\ S\end{pmatrix}A_{11}(1_r \quad T)Q^T,$$

where A_{11} is an $r\times r$ matrix of rank r, P and Q are permutation matrices, and $T = A_{11}^{-1}A_{12}$, $S = A_{21}A_{11}^{-1}$. Since

$$A_{11}^{\dagger} = A_{11}^{-1}, \qquad (1_r,0)PP^T\begin{pmatrix} 1_r\\ S\end{pmatrix}A_{11} = A_{11},$$

and

$$A_{11}(1_r \quad T)Q^TQ\begin{pmatrix} 1_r\\ 0\end{pmatrix} = A_{11},$$

it follows from Theorem 2 that

$$M^{\dagger}=Q\begin{pmatrix}1_r\\ T^*\end{pmatrix}(1_r+TT^*)^{-1}A_{11}^{-1}(1_r+S^*S)^{-1}(1_r\quad S^*)P,$$

which is a well-known expression. (See [1, pp. 193].)

(b) M has the polar decomposition

$$M=GP(=PH),$$

where P is an $m\times n$ partial isometry and G an $m\times m$ (and H an $n\times n$) Hermitian positive semidefinite matrix. (See [1, pp. 255].) Moreover, $P^{\dagger}=P^*$ and $G^{\dagger}GP=P$. Therefore, Theorem 2 can be applied:

$$\begin{aligned}(GP)^{\dagger}&=[(GP)^*GP+1-P^*P]^{-1}(GP)^*\\ &=[1+P^*(G^*G-1)P]^{-1}(GP)^*.\end{aligned}$$

If, in particular, P is nonsingular, then it follows from the definition of the Moore-Penrose inverse that $(GP)^{\dagger}=P^*G^{\dagger}$. Indeed, Theorem 2 implies

$$\begin{aligned}(GP)^{\dagger}&=(GP)^*[GP(GP)^*+1-GG^{\dagger}]^{-1}\\ &=P^*G^*[GG^*+1-GG^{\dagger}]^{-1}\\ &=P^*G^*[(GG^*)^{\dagger}+1-GG^{\dagger}]\\ &=P^*G^{\dagger}+P^*G^*-P^*G^*\\ &=P^*G^{\dagger}.\end{aligned}$$

(c) M has the singular-value decomposition

$$M=U\Sigma V^*,$$

where U and V^* are unitary matrices and $\Sigma=\operatorname{diag}(\lambda_1,\lambda_2,\dots,\lambda_r,0,\dots,0)$. Since U and V are invertible and $\Sigma^{\dagger}=\operatorname{diag}(\lambda_1^{-1},\lambda_2^{-1},\dots,\lambda_r^{-1},0,\dots,0)$, we can apply Theorem 2. It follows, by straightforward calculations, that

$$M^{\dagger} = V\Sigma^{\dagger}U^*,$$

which is a well-known fact.

(d) For any possible product of matrices A and B, the following expression is due to Cline and Greville (see [3, p. 20]):

$$(AB)^{\dagger} = (A^{\dagger}AB)^{\dagger}(ABB^{\dagger})^{\dagger}.$$

To simplify the notation, we denote $A^{\dagger}A$ by E and $BB^{\dagger}$ by F. Theorem 2 can be applied if the matrix equations $(EB)X = E$ and $Y(AF) = F$ have solutions. In this case,

$$\begin{aligned}(AB)^{\dagger} &= B^*E[EBB^*E + 1 - E]^{-1}EF[FA^*AF + 1 - F]^{-1}FA^* \\ &= B^*[EBB^*E + 1 - E]^{-1}EF[FA^*AF + 1 - F]^{-1}A^*.\end{aligned}$$

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